§1) Distributions

QFT $\to$ Q fields: self-adj operators of $\mathbb{C}$, sep, Hilbert $\mathbb{H}$
$\to$ Q states: lines through 0 of $\mathbb{H}$, (or) points in $\mathbb{P} = \mathbb{P}(\mathbb{H})$

<u>Defn:</u> (<u>Schwarz space of rapidly decreasing smooth functions</u>)

<u>$S(\mathbb{R}^n)$</u>: complex vector space of fns $f: \mathbb{R}^n \to \mathbb{C}$ with continuous partial derivative of any order for which

$$\forall p, k \in \mathbb{N} \quad |f|_{p,k} := \sup_{|\alpha| \le p} \sup_{x \in \mathbb{R}^n} |\partial^\alpha f(x)| (1 + |x|^2)^k < \infty$$

$$(|\alpha| = \alpha_1 + \cdots + \alpha_n)$$

multi-index, $\alpha = (\alpha_1, \ldots \alpha_n) \in \mathbb{N}^n$

$S(\mathbb{R}^n)$'s elements are <u>test fns</u>. $| \; |_{p,k}$ is a seminorm

---

<u>Defn:</u>

A tempered distribution T is a linear functional $T: S \to \mathbb{C}$ which is cont. wrt $| \; |_{p,k}$

Meaning: If $(f_j) \xrightarrow{\text{converges}} f \in S$ (in the sense that

$$\lim_{j \to \infty} |f_j - f|_{p,k} = 0 \quad \forall p, k \in \mathbb{N}),$$ then:

$(T(f_j)) \xrightarrow{\text{converges}} T(f)$ (in the same sense above)

Vect space of temp. dis $\equiv \mathcal{S}'(\mathbb{R}^n)$

1) Eg:-

Some distn rep. by

$$T_g(f) := \int_{\mathbb{R}^n} g(x) f(x)\,dx \;, \; f \in \mathcal{S}$$

g: measurable bounded

2) Not rep. by fn.

$$\delta_y : \mathcal{S} \longrightarrow \mathbb{C}$$
$$f \longmapsto f(y)$$

Nevertheless, sometimes, $\delta_y \equiv \delta(x-y)$ & use formal integral

$$\delta_y(f) = f(y) = \int_{\mathbb{R}^n} \delta(x-y) f(x)\,dx$$

Derivatives:

$$\frac{\partial}{\partial q^j} T(f) := -T\left(\frac{\partial}{\partial q^j} f\right)$$

Higher derivatives:

$$\partial^\alpha T(f) := (-1)^{|\alpha|}\, T(\partial^\alpha f) \;, \; f \in \mathcal{S}$$

Every distn:

$$T = \sum_{0 \le |\alpha| \le k} \partial^\alpha T_{g_\alpha} \quad (\exists\, g_\alpha : \mathbb{R}^n \to \mathbb{C})$$

PDE

$$P(X) = c_\alpha X^\alpha$$

$$P(-i\partial) = c_\alpha (-i\partial)^\alpha = \sum c_{(\alpha_1,\dots,\alpha_n)} \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$$

$$P(-i\partial)u = v$$

Eg:-

$$\Box = \partial_0^2 - \underbrace{\left(\partial_1^2 + \dots + \partial_{D-1}^2\right)}_{\Delta} = \partial_0^2 - \Delta$$

has $P = -X_0^2 + X_1^2 + \dots + X_{D-1}^2$

Fundamental soln. G is soln. to:

$$P(-i\partial)G = \delta$$

Prop.

Such a fund. soln. provides soln. to inhom. PDE $P(-i\partial)u = v$ by

$$P(-i\partial)(G * v) = T_v$$

where $(G * v)(u) := G(v * u) = G\left(\int_{\mathbb{R}^n} v(y)u(x-y)\,dy\right)$

Proof:

$$\text{Check}(\delta * v)(u) = \delta(v * u) = \delta\left(\int_{\mathbb{R}^n} v(y)u(x-y)dy\right)$$

$$= \int_{\mathbb{R}^n} v(y)u(y)dy = T_v(u)$$

Cor) $\delta * v = v$ $\qquad$ $(G * v) = T_u$ where u is the soln. to $P(-i\partial)u = v$

$\square$

Fund. soln. are obtained via Fourier transformations:

$$F(u) = \hat{u}(p) := \int_{\mathbb{R}^n} u(x)\, e^{ix\cdot p}\, dx \qquad p \in (\mathbb{R}^{1,D-1})'$$

$u: \mathbb{R}^n \to \mathbb{C}$ $\qquad$ dual space

$F(u) = \hat{u} \in \mathcal{S}$

$F: \mathcal{S} \to \mathcal{S}$ (Fourier trans. is lin, cont, invertible). We can perform Fourier transf. for distributions too in the following:

Defn: $F' : \mathcal{S}' \to \mathcal{S}'$

$T \mapsto T \circ F$

lin, cont., invertible too.

Prop

1) $F'(T_g)(v) = T_{F(g)}(v)$. Proof: Direct calculation $\square$

2) $\quad F(\partial_k u) = -ip_k F(u)$

Proof: $F(\partial_k u)(p) = \int \partial_k u(x) e^{ix\cdot p} dx \underset{\substack{\uparrow \\ \text{partial int.}}}{=} -\int u(x)\, ip_k\, e^{ix\cdot p} dp$

$= -ip_k F(u)(p)$ $\square$

III^{ly} for higher order derivatives: $F(\partial^\alpha u) = (-ip)^\alpha F(u)$

So, $P(-i\partial)u = v \Rightarrow P(p)\hat{u} = \hat{v}$ division problem for distri

$PT(u) = T(Pu)$

"Solving a division problem" is: Find a distribution T s.t. $PT = f$

Proposition:

How to determine fund. soln of $P(-i\partial)u = v$?

---

1) Solve Div. problem $PT = 1$ for T.

2) $F^{-1}(T)$

Proof:

$G = G_P$ inv. fourier transform $F^{-1}(T)$ of soln.

$PT = 1$ ie) $P\hat{G} = 1$. Claim G is fund. soln. to

$P(-i\partial)u = v$. <u>Reason</u>

$$F\left(P(-i\partial)G\right) = P(p)\hat{G} = 1$$

$$\Rightarrow P(-i\partial)G = \delta$$

<u>KG</u>

Fund. soln. of $(\Box + m^2)u = v$ (free bosonic classical particle of mass, $m>0$)

By previous result

$$(-p^2+m^2)T = 1$$

$$T = (m^2-p^2)^{-1}$$

Corres. fund. soln.

$$G(x) = (2\pi)^{-D}\int_{\mathbb{R}^D}(m^2-p^2)^{-1}e^{-ix\cdot p}\,dp$$

$\|$'ly $(\Box+m^2)\phi = 0$ solns. are

$$\phi(t,x) = (2\pi)^{D}\int_{\mathbb{R}^{D-1}} a(\mathbf{p})e^{i(\mathbf{p}\cdot x-\omega(\mathbf{p})t)} + a^*(\mathbf{p})e^{-i(\mathbf{p}\cdot x-\omega(\mathbf{p})t)}\frac{1}{2\omega(\mathbf{p})}d\mathbf{p}$$

$a, a^* \in S((\mathbb{R}^{D-1})')$ $\qquad \omega(\mathbf{p}) = \sqrt{\mathbf{p}^2+m^2}$

§2) Field operators

Denote $\mathcal{SO}(\mathbb{H})$ self adj. operators in $\mathbb{H}$

$\mathcal{O}(\mathbb{H})$ densely defined operators in $\mathbb{H}$

Defn: (Operator in $\mathbb{H}$):

Pair (A, D) $D = D_A \subset \mathbb{H}$, $\mathbb{C}$-lin. map $A : D \to \mathbb{H}$

Densely-defined operator: If D_A is dense in $\mathbb{H} \Rightarrow A$ is dense

Bounded operator: $\sup\{\|Af\| \mid f \in D_A, \|f\| \leq 1\} < \infty$

Closed operator: An operator B in $\mathbb{H}$ is closed if its graph $\Gamma(B) = \{(f, B(f)) \mid f \in D_B\} \subset \mathbb{H} \times \mathbb{H}$ is closed where,

$\mathbb{H} \times \mathbb{H} \simeq \mathbb{H} \oplus \mathbb{H}$ & it's Hilbert space structure is given by the inner prd:

$$\langle (f, f'), (g, g') \rangle = \langle f, g \rangle + \langle f', g' \rangle$$

Adjoint of an operator:

Every densely defined operator on $\mathbb{H}$ has an "adjoint" A^* defined in the following way:

For A, define A^* with

$$D_{A^*} = \{ f \in \mathbb{H} \mid \exists h \in \mathbb{H} \ \forall g \in D_A : \langle h, g \rangle = \langle f, Ag \rangle \} \text{ and}$$

$$\langle A^* f, g \rangle := \langle f, Ag \rangle, \ f \in D_{A^*}, \ g \in D_A$$

Prop:

A is Dense + Bounded $\Rightarrow$ A is cont. & unique linear cont. continuation to all of $\mathbb{H}$ exists.

Self-adjoint operator:

A is a self-adjoint operator if $D_A = D_{A^*}$ and $A^* f = Af$ for all $f \in D_A \ (= D_{A^*})$

Symmetric operator:

An operator, A which is densely defined s.t.

$$\langle Af, g \rangle = \langle f, Ag \rangle, \ f, g \in D_A$$

Prop:-

Self-adjoint operators are trivially symmetric and closed (closed since adjoint operators are closed in general)

Defn. (Spectrum of a closed operator):

For a closed operator, A denote

$$\sigma(A) = \left\{ \lambda \in \mathbb{C} \;\middle|\; \begin{array}{l} (A - \lambda \, id_{\mathbb{H}})^{-1} \text{ does not exist as a} \\ \text{bounded operator} \end{array} \right\}$$

called Spectrum of A.

Prop:

1) For a closed operator A, $\sigma(A)$ is a closed subset of $\mathbb{C}$

2) For a self-adj. operator A, $\sigma(A)$ is completely contained in $\mathbb{R}$

Prop:

For a self-adj. operator A, there exists a unique representation $U: \mathbb{R} \longrightarrow U(\mathbb{H})$ satisfying

$$\lim_{t \to 0} \frac{U(t)f - f}{t} = -iAf$$

for each $f \in D_A$. Notation: U is denoted $U(t) = e^{-itA}$ and A is called "infinitesimal generator of $U(t)$"

Converse of above prop:

Thm: (Stone's thm)

Let U be a unitary representation of $\mathbb{R}$ in the Hilbert space, $\mathbb{H}$. Then the operator A defined by:

$$Af := \lim_{t \to 0} i \left(\frac{U(t) f - f}{t} \right)$$

in the domain in which this limit exists wrt norm of $\mathbb{H}$, is self-adjoint & "generates" (in the prev. proposition's sense) U. i.e.) $U(t) = e^{-itA}$, $t \in \mathbb{R}$.

Field operators:

Analogue of classical fields in QFT. You need operator valued distributions to describe quantum fields (and not merely a map from the manifold $M = \mathbb{R}^{1,D-1}$ to SO) because:

In classical field theories, the Poisson bracket of a field ϕ at points $x, y \in M$ with $x^0 = y^0$ (equal time) is of the form

$$\{\phi(x), \phi(y)\} = \delta(\underline{x} - \underline{y})$$

where $\left.\begin{array}{l} \underline{x} = (x^1, \ldots, x^{D-1}) \\ \underline{y} = (y^1, \ldots, y^{D-1}) \end{array}\right\}$ the space parts of x & y.

But this can only be described by operator valued distributions (due to right side of the eqn) rather than a mere operator valued map.

Defn:

A field operator (or) quantum field is defined as an operator-valued distribution ie)

$$\Phi : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{O}$$

s.t. there exists a dense subspace $D \subset \mathbb{H}$ satisfying

1) For each $f \in \mathcal{S}$, the domain of defn. $D_{\Phi(f)}$ contains D.

2) The induced map $\mathcal{S} \to \mathrm{End}(D)$ is linear
$$f \longmapsto \Phi(f)|_D$$

3) For each $v \in D$, $w \in \mathbb{H}$, the assignment $f \longmapsto \langle w, \Phi(f)(v) \rangle$ is a tempered distribution

§3) Wightman axioms

Relativistic invariance:

$M = \mathbb{R}^{1,3}$ (or sometimes $\mathbb{R}^{1,D-1}$) with Lorentz metric

$$x^2 = \langle x, x \rangle = x^0 x^0 - \sum_{j=1}^{D-1} x^j x^j \quad (x = (x^0, \dots, x^{D-1}) \in M)$$

Defn:

Two subsets $X, Y \subset M$ are called <u>space-like separated</u> if for any $x \in X$ and $y \in Y$, the condition $(x-y)^2 < 0$ is satisfied, that is

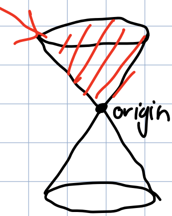
$$(x^0 - y^0)^2 < \sum_{j=1}^{D-1} (x^j - y^j)^2$$

<u>Defn:</u>

The <u>forward cone</u> $C_+ := \{ x \in M \mid x^2 = \langle x, x \rangle \geq 0, \; x^0 \geq 0 \}$

$\downarrow$

Time-like separated from origin:



<u>Causal order:</u> $(x \geq y) \iff (x - y \in C_+)$

Relativistic invariance of classical point particles in $\mathbb{R}^{1,D-1}$ is described by Poincaré grp $P := (1, D-1)$

$$P \cong L \ltimes \mathbb{R}^n$$

$(L = \text{Lorentz grp}: SO_0(1, D-1) \subset GL(D, \mathbb{R}))$

identity component

P preserves causal structure & space-like separatedness

P acts on $\mathcal{S}(\mathbb{R}^D)$ as:

$$\left\{ h.f(x) := f(h^{-1}x) \quad \text{with} \quad g.(h.f) = (g.h).f \right.$$

this action is continuous.

Elements of P are written as: (q, Λ) where $q \in M$, $\Lambda \in L$. ie)

$$(q, \Lambda) f(x) = f(\Lambda^{-1}(x - q))$$

The relativistic invariance of a quantum system wrt Minkowski space is in general given by a projective rep $P \longrightarrow U(\mathbb{P}(\mathbb{H}))$ of the Poincaré grp, P in the space of states, $\mathbb{P}(\mathbb{H})$ of the quantum system.

(By Thm-4.8, we can lift this rep. uniquely to a unitary rep of the double cover of P. This covering is also called P)

Unitary rep. of Poincaré grp P

$$U : P \longrightarrow U(\mathbb{H})$$

$$(q, \Lambda) \longmapsto U(q, \Lambda)$$

Since $\mathbb{R}^{1,D-1} \subset \mathcal{P}$ is abelian, we can apply Stone's thm component-wise to obtain the restriction of U to M in the form

$$U(q, 1) = \exp(iqP) = e^{i(q^0 P_0 - q^1 P_1 - \cdots - q^{D-1} P_{D-1})}$$

$q \in \mathbb{R}^{1,D-1}$

$P_0, \ldots, P_{D-1}$ are self-adj. operators on $\mathbb{H}$.

P_0 is interpreted as energy operator & P_j $(j > 0)$ as components of momentum.

Wightman axioms:

A Wightman QFT (WQFT) in D dimensions consists of:

1) Space of states i.e.) projective space $\mathbb{P}(\mathbb{H})$ of a separable complex Hilbert space $\mathbb{H}$

2) Vacuum vector $\Omega \in \mathbb{H}$ of norm 1

3) A unitary rep $U : \mathcal{P} \longrightarrow U(\mathbb{H})$ of Poincaré grp $\mathcal{P}$ and of the covering grp. of Poincaré grp

4) A collection of field operators Φ_a $(a \in I)$

$$\Phi_a : \mathcal{S}(\mathbb{R}^D) \longrightarrow \mathcal{O}$$

with a dense subspace $D \subset \mathbb{H}$ as their common domain (ie) $D_{\Phi_a(f)}$ contains D for all $a \in I$, $f \in \mathcal{S}$) s.t. Ω is in D.

The above data should satisfy the following four axioms:

Axiom (WI) : Covariance

1) $\left.\begin{array}{l} U(q,\Lambda)\Omega = \Omega \\ U(q,\Lambda) D \subset D \end{array}\right\}$ for all $(q,\Lambda) \in P$

2) $D \subset \mathbb{H}$ is invariant in the sense $\Phi_a(f) D \subset D$ $\forall\, f \in \mathcal{S},\ a \in I$

3) On D,

$$U(q,\Lambda)\,\Phi_a(f)\,U(q,\Lambda)^* = \Phi_a((q,\Lambda)f)$$

$\forall\, f \in \mathcal{S},\ (q,\Lambda) \in P$

(Actions on $\mathbb{H}$ and $\mathcal{S}$ are equivariant & P acts on $\mathrm{End}(D)$ by conjugation)

Axiom (W2): Locality

$\Phi_a(f)$ and $\Phi_b(g)$ commute on $\mathcal{D}$ if the supports of f and $g \in \mathcal{S}$ are space-like separated.

i.e) $\Phi_a(f)\Phi_b(g) - \Phi_b(g)\Phi_a(f) =: [\Phi_a(f), \Phi_b(g)] = 0$

Axiom (W3): Spectrum condition:

The joint spectrum of the operators P_j is contained in the forward cone C_+

Axiom (W4): Uniqueness of vacuum:

The only vectors in $\mathbb{H}$ left invariant by the translations $U(a,1)$, $a \in M$ are scalar multiples of Ω

Remarks:

1) For real valued $f \in \mathcal{S}$, $\Phi_a(f)$ should be essentially self adjoint (there exists a self adjoint operator which restricts to $\Phi_a(f)$)

2) Axiom (W1) is for scalar fields which transform under trivial rep of L. If fields transform according to non-trivial, fin. dim complex or real rep $R: L \longrightarrow GL(W)$ of double cover of L, then (W2) has to be replaced by

$$U(q,\Lambda)\,\Phi_j(f)\,U(q,\Lambda)^* = \sum_{k=1}^{m} R_{jk}(\Lambda^{-1})\,\Phi_k((q,\Lambda)f)$$

- W is identified with $\mathbb{R}^m$ (or $\mathbb{C}^m$)
- $R(\Lambda)$ matrices $(R_{jk}(\Lambda))$
- Φ_a are components of fields which can be grouped together to a vector $(\Phi_1, \dots \Phi_m)$

3) The axiom (W2) describes only bosonic fields. (For fermionic case, read Chapter-10)

4) Axiom (W3) $\Rightarrow$ eigenvalues p_ε of P_ε grouped into $p = (p_0, \dots, p_{D-1})$ satisfies $p \in C_+$. (?) with P_0 as energy operator, this says that the system has no negative energy states

5) $P^2 = P_0^2 - P_1^2 - \dots - P_{D-1}^2$ is mass squared operator with condition $p^2 \geq 0$ for each D-tuple of

eigenvalues p_ξ of P_ξ if (W3) is satisfied.

6) In addition to (W1) — (W4), in many cases the following completeness condition is added:

Subspace $D_0 \subset D$ spanned by all vectors

$$\Phi_{a_1}(f_1)\,\Phi_{a_2}(f_2)\cdots\Phi_{a_m}(f_m)\Omega$$

is dense in D and thus dense in $\mathbb{H}$

<u>Example: (Free Bosonic QFT)</u>

Construct a WQFT for a quantized boson of mass $m>0$ in $D=4$ (3 dimensional space). We "define"/seek a quantized boson of mass $m>0$ in $D=4$ as a field operator which satisfies the following properties:

$$\Phi : \mathcal{S}(\mathbb{R}^4) \longrightarrow \mathcal{SO}(\mathbb{H})$$ on a Hilbert space, $\mathbb{H}$ s.t. $\forall f, g \in \mathcal{S}$,

1) Φ satisfies

$$\Phi(\Box f + m^2 f) = 0 \quad \forall f \in \mathcal{S}$$

2) Φ obeys the commutation relation

$$[\Phi(f), \Phi(g)] = -i \int_{\mathbb{R}^4 \times \mathbb{R}^4} f(x) D_m(x-y) g(y)\, dx\, dy$$

where $D_m(x) := i(2\pi)^{-3} \int_{\mathbb{R}^D} \operatorname{sgn}(p_0)\, \delta(p^2 - m^2)\, e^{-ip\cdot x}\, dp$

The construction of such a field & corresponding Hilbert space, $\mathbb{H}$ (which obeys WQFT axioms) is a Fock space construction. Let's construct $\mathbb{H}$ first:

$H_1 = \mathcal{S}(\Gamma_m) \cong \mathcal{S}(\mathbb{R}^3)$ where

$$\Gamma_m = \{ p \in (\mathbb{R}^{1,D-1})' \mid p^2 = m^2,\ p_0 > 0 \}$$

The isomorphism is induced by the global chart

$$\begin{aligned} \eta : \mathbb{R}^3 &\longrightarrow \Gamma_m \\ \vec{p} &\longmapsto (\omega(\vec{p}), \vec{p}) \end{aligned} \quad \text{where } \omega(\vec{p}) = \sqrt{\vec{p}^2 + m^2}$$

H_1 is dense in $\mathbb{H}_1 := L^2(\Gamma_m, \underbrace{d\lambda_m}_{\text{Lorentz-invariant measure}}) =$ complex Hilbert space of square integrable fns on Γ_m

Let H_N denote the space of rapidly decreasing fns. on N-fold prod of Γ_m which are symmetric in variable $(\vec{p}_1, \ldots, \vec{p}_N) \in \Gamma_m^N$. H_N has the inner prod:

$$\langle u,v\rangle := \int_{\Gamma_m^N} \bar{u}(\xi_1,\dots,\xi_N)\, v(\xi_1,\dots,\xi_N)\, d\lambda_m(\xi_N)$$

where points in Γ_m are denoted by ξ_j (or) ξ

Denote the direct sum

$$D := \bigoplus_{N=0}^{\infty} H_N \qquad (H_0 = \mathbb{C},\ \Omega := 1 \in H_0)$$

has a inner prod:

$$\langle f,g\rangle := \bar{f}_0 g_0 + \sum_{N\geq 1} \frac{1}{N!} \langle f_N, g_N\rangle$$

where $\left.\begin{array}{l} f=(f_0,f_1,\dots) \\ g=(g_0,g_1,\dots) \end{array}\right\} \in D$

Completion of D wrt the above inner prod. is denoted by $\mathbb{H}$, the Fock space, the desired Hilbert space.

Now let's construct the fields:

The operators $\Phi(f)$, $f \in S$ defined on $g=(g_0,g_1,\dots) \in D$ by:

$$\left(\Phi(f)g\right)_N(\xi_1,\dots,\xi_N) := \int_{\Gamma_m} \overset{\swarrow \text{Fourier trans.}}{\hat{f}}(\xi)\, g_{N+1}(\xi,\xi_1,\dots,\xi_N)\, d\lambda_m(\xi)$$

$$+ \sum_{j=1}^{N} \hat{f}(-\xi_j)\, g_{N-1}(\xi_1,\dots,\cancel{\xi_j},\dots,\xi_N)$$

where $\cancel{\xi_j}$ means that this variable is omitted.

The above $\mathbb{H}$ & the quantum field satisfies WQFT axioms & indeed is a WQFT now. (Checking is direct verification which is omitted here).

Another reason why we describe fields as operator valued distributions & not merely operator valued maps:

Prop:

Let Φ be a field in a WQFT which can be realized as a map $\Phi : M \longrightarrow \mathcal{O}$ where Φ^* belong to the fields of the WQFT too. Then $\Phi(x) = c\Omega$ is the constant operator for some $c \in \mathbb{C}$ (ie) nothing interesting happens).